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TORSION OF PRISMATIC ELASTIC BODIES CONTAINING SCREW DISLOCATIONS†

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The problem of the stressed state of a prismatic anisotropic rod containing screw dislocations, the axes of which are parallel to the rod axis, is considered. Such defects may arise during the growth of filamentary crystals (metal "whiskers"), and may also exist in multiply connected cylindrical structures. The torsion of an anisotropic elastic bar with a multiply connected cross-section is investigated initially, assuming that the stresses and strains are single-valued but dispensing with the requirement that the warping function should be single-valued. The boundary-value problem is formulated in terms of the Prandtl stress function, which, unlike the warping function, is single-valued in a multiply connected region. A variational formulation of the boundary-value problem for the stress function is given. From the variational principle obtained a torsion boundary-value problem is formulated when there are lumped or continuously distributed dislocations. A modification of the membrane analogy for the torsion problem is proposed which takes into account the presence of dislocations. General theorems of the theory of the torsion of a rod containing dislocations are formulated. An effective formula is derived for the angle of torsion of a bar due to a specified dislocation distribution. Problems on dislocations in a thin-walled rod and a rectangular anisotropic bar are solved. © 2002 Elsevier Science Ltd. All rights reserved.

1. FUNDAMENTAL RELATIONS OF THE THEORY OF THE TORSION OF AN ANISOTROPIC BAR

The system of equations describing the torsion of a prismatic body of anisotropic material, which possesses a plane of elastic symmetry, orthogonal to the bar axis, has the form [1]

$$\frac{\partial\tau_{13}}{\partial x_1}+\frac{\partial\tau_{23}}{\partial x_2}=0 \tag{1.1}$$

$$\gamma_{13}=k\tau_{13}+l\tau_{23},\quad \gamma_{23}=l\tau_{13}+m\tau_{23} \tag{1.2}$$

$$\gamma_{13}=\frac{\partial w}{\partial x_1}-\omega x_2,\quad \gamma_{23}=\frac{\partial w}{\partial x_2}+\omega x_1 \tag{1.3}$$

$$k=\frac{1}{G_{23}},\quad m=\frac{1}{G_{13}},\quad l=\frac{\mu_{31,12}}{G_{23}}=\frac{\mu_{23,31}}{G_{13}}$$

Here x_1 and x_2 are Cartesian coordinates in the cross-section plane of the bar, τ_{13} and τ_{23} are shear stresses, γ_{13} and γ_{23} are the components of the shear deformation, ω is the angle of torsion, $w(x_1, x_2)$ is the warping function, k, l and m are the elastic compliances, G_{13} and G_{23} are the shear moduli, and $\mu_{31,12}$ and $\mu_{23,31}$ are the Chentsov coefficients [1]. Introducing the coordinate unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3 = \mathbf{i}_1\times\mathbf{i}_2$, the vector of the shear stresses $\boldsymbol{\tau} = \tau_{13}\mathbf{i}_1 + \tau_{23}\mathbf{i}_2$, the shear deformation vector $\boldsymbol{\gamma} = \gamma_{13}\mathbf{i}_1 + \gamma_{23}\mathbf{i}_2$ and the compliance tensor $\boldsymbol{\lambda} = k\mathbf{i}_1\mathbf{i}_1 + l(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1) + m\mathbf{i}_2\mathbf{i}_2$, Eqs (1.1)–(1.3) can be represented in the invariant form

$$\operatorname{div}\boldsymbol{\tau}=0 \tag{1.4}$$

$$\boldsymbol{\gamma}=\boldsymbol{\lambda}\cdot\boldsymbol{\tau} \tag{1.5}$$

$$\boldsymbol{\gamma}=\operatorname{grad}w-\omega\mathbf{e}\cdot\mathbf{r} \tag{1.6}$$
$$\mathbf{e}=-\mathbf{i}_3\times\mathbf{E},\quad \mathbf{r}=x_1\mathbf{i}_1+x_2\mathbf{i}_2$$

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Here div and grad are the plane divergence and gradient operators, $\mathbf{E}$ is the unit tensor, $\mathbf{e}$ is the discriminant tensor and $\mathbf{r}$ is the radius vector of a point of the cross-section.

The equilibrium equations (1.4) are satisfied identically after introducing the Prandtl stress function F

$$\boldsymbol{\tau} = \mathbf{e} \cdot \operatorname{grad} F \tag{1.7}$$

while elimination of the warping function w from relation (1.6) leads to the deformation compatibility equation

$$\operatorname{div}(\mathbf{e} \cdot \boldsymbol{\gamma}) = 2\omega \tag{1.8}$$

From (1.5), (1.7) and (1.8) we obtain the following equation for the stress function

$$\operatorname{div}(\mathbf{L} \cdot \operatorname{grad} F) = -2\omega \tag{1.9}$$

$$\mathbf{L} = -\mathbf{e} \cdot \boldsymbol{\lambda} \cdot \mathbf{e} = m\mathbf{i}_1\mathbf{i}_1 - l(\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_1) + k\mathbf{i}_2\mathbf{i}_2$$

The boundary condition for the function F on the boundary of the region σ, occupied by the cross-section of the rod, expresses the fact that there is no load on the side surface of the prismatic body and, by (1.7), has the form

$$\mathbf{n} \cdot \boldsymbol{\tau} = \partial F/\partial s = 0 \tag{1.10}$$

where $\mathbf{n}$ is the unit normal to $\partial\sigma$ and s is the length of the arc of the plane curve $\partial\sigma$.

2. GENERALIZATION OF BREDT'S THEOREM

Suppose the bar cross-section σ is a multiply connected region, homomorphic to a circle with circular apertures. The external contour of the region σ will be denoted by Γ_0, while the contours of the apertures will be denoted by $-\Gamma_t (t = 1, 2, \ldots, N)$. By virtue of condition (1.10), the stress function F is single-valued in the region σ and takes constant values C_0 and C_t on each of the closed curves Γ_0 and Γ_t. Since the addition of an arbitrary constant to the function F has no effect on the stressed state of the bar, without loss of generality we can put $C_0 = 0$, which leads to the well-known boundary conditions [2] in the problem of the torsion of a multiply connected cylindrical body

$$F\big|_{\Gamma_0} = 0, \quad F\big|_{\Gamma_t} = C_t, \quad t = 1, 2, \ldots, N \tag{2.1}$$

It does not follow from the single-valuedness of the stress field τ and the strain field γ in the multiply connected region that the field of the axial displacements $w(r)$ will also be single-valued. As follows from equality (1.6), the warping is expressed by the formula

$$w(\mathbf{r}) = \int_{\mathbf{r}_0}^{\mathbf{r}} d\mathbf{r} \cdot (\boldsymbol{\gamma} + \omega \mathbf{e} \cdot \mathbf{r}) + w(\mathbf{r}_0) \tag{2.2}$$

When the compatibility equation (1.8) is satisfied, the curvilinear integral in (2.2) will be independent of the integration path if the region σ is simply connected. In the case of a multiply connected region, expression (2.2), generally speaking, gives a multivalued function. The non-uniqueness can be eliminated by converting the region σ into a simply connected region by carrying out the necessary number of branch cuts. Then the values of the function $w_\pm$ on the opposite edges of each branch cut can differ by a constant quantity $w_+ - w_- = b_t$. By (2.2) the constants b_t are independent of the choice of the system of branch cuts and can be expressed in terms of the strain field as follows:

$$b_t = \oint_{\Gamma_t} (\boldsymbol{\gamma} - \omega \mathbf{r} \cdot \mathbf{e}) \cdot d\mathbf{r} = \oint_{\Gamma_t} \boldsymbol{\gamma} \cdot d\mathbf{r} - 2\omega S_t \tag{2.3}$$

where S_t is the area of the tth aperture.

The fact that the constants b_t are non-zero indicates the existence of screw dislocations in the multiply connected cylinder, i.e. translational Volterra dislocations with Burger's vectors parallel to the generatrix of the cylinder and having a length b_t. From relations (1.5), (1.7) and (2.3) we obtain

$$\oint_{\Gamma_t} \mathbf{n}\cdot\mathbf{L}\cdot \operatorname{grad} F ds + b_t + 2\omega S_t = 0, \quad t = 1,2,\ldots,N \tag{2.4}$$

Here $\mathbf{n}$ is the outward normal to the region occupied by the aperture. Relations (2.4) serve to determine the unknown constants C_t and are a generalization of Bredt's theorem [2] on the circulation of shear stresses, taking into account screw dislocations and the anisotropy of the material.

The torque, which must be applied to the ends of the multiply connected bar, in order to ensure a specified angle of torsion ω, can be expressed by the formula [2]

$$M = \mathbf{i}_3 \cdot \iint_\sigma \mathbf{r}\times\boldsymbol{\tau} d\sigma = 2\iint_\sigma F d\sigma + 2\sum_{t=1}^{N} C_t S_t \tag{2.5}$$

3. THE MEMBRANE ANALOGY WHEN THERE ARE DISLOCATIONS

It is well known [2] that the Prandtl membrane analogy for the torsion problem can be modified by taking into account the presence of dislocations and the anisotropy of the material. We will consider an extremely thin elastic plate (a membrane), in which a uniform plane stressed state is produced, described by a stress tensor, which, apart from a constant factor, is identical with the compliance tensor $\mathbf{L}$ from Eq. (1.9). Since the specific strain energy of the elastic body $^1/_2\,\tau\cdot\gamma$ is positive, the tensors λ and $\mathbf{L}$ are positive definite, which implies the inequalities $k > 0$, $m > 0$ and $km - l^2 > 0$. We obtain the spectral expansion of the tensor $\mathbf{L}$

$$\mathbf{L} = L_1\mathbf{d}_1\mathbf{d}_1 + L_2\mathbf{d}_2\mathbf{d}_2 \tag{3.1}$$

$$L_{1,2} = \frac{1}{2}(k+m) \pm \frac{1}{2}\sqrt{(k-m)^2 + 4l^2}$$

$$\mathbf{d}_1 = \mathbf{i}_1\cos\varphi + \mathbf{i}_2\sin\varphi, \quad \mathbf{d}_2 = -\mathbf{i}_1\sin\varphi + \mathbf{i}_2\cos\varphi$$

$$\operatorname{tg}\varphi = \frac{l}{k - L_1} = \frac{2l}{k - m - \sqrt{(k-m)^2 + 4l^2}}$$

Here L_1 and L_2 are positive eigenvalues, while $\mathbf{d}_1$ and $\mathbf{d}_2$ are unit orthogonal eigenvectors of the tensor $\mathbf{L}$. By expression (3.1), the stress state of the membrane, corresponding to the tensor $\mathbf{L}$, can be produced by a tension L_1 in the direction $\mathbf{d}_1$ and a tension L_2 in the orthogonal direction $\mathbf{d}_2$. A membrane stretched in this way is clamped along the contour Γ_0 and loaded by a transverse uniform pressure p, proportional to the angle of torsion ω. The equation for the sag of the membrane u can be obtained from the equation of the sag of a prestressed plate [3], by allowing its cylindrical stiffness to approach zero. As a result, for the case of a simply connected region we obtain a boundary-value problem identical with the problem for the stress function F

$$\operatorname{div}(\mathbf{L}\cdot\operatorname{grad} u) = -p, \quad u|_{\Gamma_0} = 0 \tag{3.2}$$

To model a multiply connected cylindrical body without dislocations, rigid horizontal discs, for which only translational vertical displacements are permitted, are attached to the stretched membrane in the regions bounded by the contours Γ_t. After this the whole system is loaded with a uniform pressure p. The conditions of equilibrium of all the forces applied to each disc correspond to Eqs (2.4) when $b_t = 0$ in the membrane analogy [2].

If dislocations are present in a multiply connected cylinder, then, by (2.4), an additional point force, coinciding (apart from a certain dimensional factor) with the length of the Burgers vector corresponding to this dislocation, must be applied to each disc. Hence, the existence of screw dislocations in a multiply connected cylinder is modelled in the membrane analogy by point forces applied to rigid discs.

The main value of the membrane analogy described is the fact that it can be used to represent clearly the solution of boundary-value problem (1.9), (2.1), (2.4) for the stress function F. In this case there is no need in fact to carry out an experiment with a loaded membrane. A thought experiment is quite sufficient.

4. THE VARIATIONAL PRINCIPLE

The specific additional work of an anisotropic body in the torsion problem can be expressed as follows based on relations (1.5) and (1.7)

$$V = \frac{1}{2}\boldsymbol{\gamma}\cdot\boldsymbol{\tau} = \frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{\lambda}\cdot\boldsymbol{\tau} = \frac{1}{2}\operatorname{grad} F\cdot\mathbf{L}\cdot\operatorname{grad} F \tag{4.1}$$

We will prove that boundary-value problem (1.9), (2.1), (2.4) is equivalent to the variational problem of the minimum of the functional

$$\Pi[F, C_1, ..., C_W] = \iint\limits_{\sigma} (V - 2\omega F)d\sigma - \sum_{t=1}^{N} (2\omega S_t + b_t)C_t \tag{4.2}$$

The functional Π is defined in the set of twice differentiable stress functions satisfying boundary conditions (2.1). The constants $C_t (t = 1, \dots, N)$, on which the functional depends, is unknown in advance and must be varied. Using expression (4.1), we calculate the variation of functional (4.2)

$$\delta\Pi = \iint\limits_{\sigma} (\operatorname{grad} F\cdot\mathbf{L}\cdot\operatorname{grad}\delta F - 2\omega\delta F)d\sigma - \sum_{t=1}^{N} (2\omega S_t + b_t)\delta C_t = \tag{4.3}$$

$$= \iint\limits_{\sigma} [\operatorname{div}(\delta F\mathbf{L}\cdot\operatorname{grad} F) - \delta F\operatorname{div}(\mathbf{L}\cdot\operatorname{grad} F) - 2\omega\delta F]d\sigma - \sum_{t=1}^{N} (2\omega S_t + b_t)\delta C_t$$

Using Green's formula and taking conditions (2.1) into account, we obtain

$$\delta\Pi = -\iint\limits_{\sigma} [\operatorname{div}(\mathbf{L}\cdot\operatorname{grad} F) + 2\omega]\delta F d\sigma - \sum_{t=1}^{N} \left(\oint\limits_{\Gamma_t} \mathbf{n}\cdot\mathbf{L}\cdot\operatorname{grad} F ds + 2\omega S_t + b_t \right)\delta C_t \tag{4.4}$$

As can be seen from expression (4.4), the necessary and sufficient conditions for the functional Π to be stationary consist of Eqs (1.9) and relations (2.4). The property that the functional should be a minimum at a stationary point follows from the fact that the tensor **L** is positive definite. The variational formulation of the torsion problem was presented previously in [2] for an isotropic material ignoring dislocations.

We will use the variational principle to solve the problem of the torsion of a thin-walled tube containing a dislocation. The section of the rod in this case is a doubly connected region, where the contours Γ_0 and Γ are close to one another. A screw dislocation in such a body can be produced by cutting the tube along the generatrix, a longitudinal shift of the edge of the cut by a distance b_1 and cementing them in the new position. The defect described can arise in the structure when it is being manufactured. We will assume that the curve Γ_1 is given by the equations $x_1 = x_1(s)$ and $x_2 = x_2(s)$, where s is the current length of the arc. A position of a point of the region σ will be specified by the coordinates s and ζ, where ζ is the distance measured along the normal to Γ_1, where $\zeta = 0$ on Γ_1 and $\zeta = h$ on Γ_0. We will assume the wall thickness h to be constant. Basing ourselves on the membrane analogy, we will approximate the stress function in the case of a thin wall by an expression which satisfies conditions (2.1)

$$F = C_1(1 - \zeta/h) \tag{4.5}$$

Taking expression (4.5) into account we will write the functional of the additional work (4.2) in the form

$$\Pi = \frac{Q}{2h}C_1^2 - \omega L_1 h C_1 - (2\omega S_1 + b_1)C_1 \tag{4.6}$$

$$Q = \oint\limits_{\Gamma_1} \left[m\left(\frac{dx_2}{ds}\right)^2 + 2l\frac{dx_1}{ds}\frac{dx_2}{ds} + k\left(\frac{dx_1}{ds}\right)^2 \right] ds$$

(L_1 is the perimeter of the contour Γ_1). By expression (4.6) we find from the stationarity condition $\partial\Pi/\partial C_1 = 0$

$$QC_1 = (2S_1 + L_1 h)\omega h + b_1 h \tag{4.7}$$

The torque is found from formulae (2.5) and (4.5)

$$M = (2S_1 + L_1 h)C_1 \tag{4.8}$$

From relations (4.7) and (4.8) we find the angle of torsion ω which occurs when there is no torque, i.e. when $M = 0$

$$\omega = -\frac{b_1}{2S_1^*}, \quad S_1^* = S_1 + \frac{L_1 h}{2}. \tag{4.9}$$

In (4.9) S_1^* is the area bounded by the middle line of the cross-section, i.e. by the contour passing through the middle of the contours Γ_0 and Γ_1. By relations (4.5) and (4.8), we have $F = 0$ when $M = 0$. This means that when there are no external loads, a screw dislocation in a thin-walled rod of arbitrary closed profile produces a torsion of the rod, but produces no stresses in it. As follows from expression (4.9), the value of the torsion is independent of the physical properties of the material, and is defined purely by the geometrical characteristics of the cross-section.

We will consider one more example of a pentaconnected thin-walled tube, the cross-section of which is a thin circular ring of radius r_0 with two orthogonal diametral partitions (Fig. 1). The thickness of the ring and the partitions is h.

We will denote the value of the stress function in the fourth ring, adjoining the sth aperture, by $F_s (s = 1, 2, 3, 4)$, and on the part of the partition separating the sth and tth apertures by F_{st}. Assuming that $h \ll r_0$, we use the approximation of the stress function which satisfies boundary conditions (2.1)

$$\begin{aligned} &F_s = C_s h^{-1}(r - r_0), \quad r_0 - h \leqslant r \leqslant r_0 \\ &2F_{12} = C_1\zeta_1^+ + C_2\zeta_1^-, \quad 2F_{23} = C_2\zeta_2^+ + C_3\zeta_2^- \\ &2F_{34} = C_3\zeta_1^+ + C_4\zeta_1^-, \quad 2F_{41} = C_1\zeta_2^+ + C_4\zeta_2^- \\ &\zeta_{1,2}^\pm = 1 \pm 2x_{1,2}/h \end{aligned} \tag{4.10}$$

(r is the radial coordinate of an arbitrary point of the ring). Assuming the material of the tube to be isotropic with shear modulus G, using relations (4.2) and (4.10) we obtain the following expression for the functional of the additional work

$$\Pi = \sum_{t=1}^{4}\left[\frac{(\pi+4)r_0}{4Gh}C_t^2 - \left(3\omega h r_0 + \frac{\pi}{2}\omega r_0^2 + b_t\right)C_t\right] \tag{4.11}$$

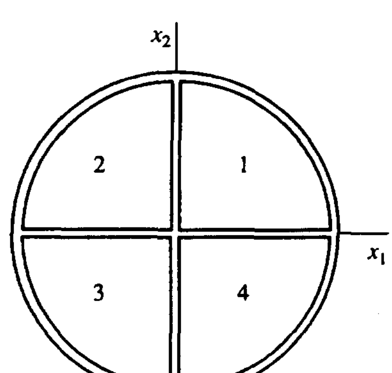

Fig. 1

From the conditions for functional (4.11) to be stationary $\partial\Pi/\partial C_t = 0$ when $M = 0$ we obtain

$$\omega = -\frac{b_1 + b_2 + b_3 + b_4}{2(6r_0 h + \pi r_0^2)},$$

$$C_s = \frac{Gh(4b_s - \sum_{t=1}^{4} b_t)}{2r_0(4+\pi)}, \quad s = 1,\ 2,\ 3,\ 4 \tag{4.12}$$

5. LUMPED AND CONTINUOUSLY DISTRIBUTED DISLOCATIONS

Using the variational principle, we will change from a multiple connected bar with isolated dislocations to the case of a simply connected cylinder with a discrete set of lamped dislocations. To do this we will reduce without limit the diameters of the apertures, contracting each contour Γ_t $(t = 1, \ldots, N)$ to a certain point r_t. The length of the Burgers vector b_t remains unchanged on taking this limit. Since the values of the stress function F are the same at all points of the contour Γ_t, in the limit the constants C_t $(t = 1, \ldots, N)$ are identical with the values of the stress function at the points r_t, the areas S_t vanish, and functional (4.2) will have the following expression

$$\Pi[F] = \iint_\sigma (V - 2\omega F)d\sigma - \sum_{t=1}^{N} b_t F(\mathbf{r}_t) = \iint_\sigma (V - 2\omega F)d\sigma - \iint_\sigma \beta^* F d\sigma \tag{5.1}$$

$$\beta^* = \sum_{t=1}^{N} b_t \delta(\mathbf{r} - \mathbf{r}_t)$$

where $\delta(\mathbf{r}-\mathbf{r}_t)$ is the delta function of two variables.

In the membrane analogy this passage to the limit leads to a simply connected membrane, loaded not only with a uniform pressure but also with point forces b_t, applied at the points r_t.

If the number of concentrated dislocations, situated in a certain part of the region σ, is extremely large, it is best to change to a continuous dislocation distribution. To do this it is sufficient, in expression (5.1), to replace the generalized function β^* by the usual distribution and call it the screw dislocation density β. The physical meaning of the dislocation density is that the overall Burgers vector $\mathbf{B}$ of all the dislocations contained in a certain subregion $\sigma' \subset \sigma$ is calculated from the formula

$$\mathbf{B} = \mathbf{i}_3 \iint_{\sigma'} \beta d\sigma$$

The functional of the additional work Π and the equation for the stress function when there are continuously distributed dislocations, which follows from the variational principle $\delta\Pi = 0$, have the form

$$\Pi = \iint_\sigma (V - 2\omega F - \beta F)d\sigma \tag{5.2}$$

$$\operatorname{div}(\mathbf{L} \cdot \operatorname{grad} F) = -2\omega - \beta(x_1, x_2) \tag{5.3}$$

In the membrane analogy, the continuous dislocation field, by Eq. (5.3), is modelled by the application of a variable normal load, the density of which is proportional to the dislocation density β.

Note that in the same region σ, where $\beta(x_1, x_2) \neq 0$, the warping function $w(x_1, x_2)$ does not exist, since the compatibility equation (1.8) is not satisfied. This prevents the existence of elastic displacements in the cross-section plane, so that the constant ω preserves its meaning of the angle of torsion for continuously distributed screw dislocations also.

Continuously distributed defects may also exist in a multiply connected cylindrical body containing Volterra dislocations. The functional of additional work for this general case, by virtue of (4.2) and (5.2), can be written in the form

$$\Pi[F] = \iint_\sigma (V - 2\omega F - \beta F)d\sigma - \sum_{t=1}^{N} (2\omega S_t + b_t)C_t \tag{5.4}$$

Because one can assume the density β to be a generalized function, expression (5.4) is also valid when there are lumped dislocations.

As an example we will consider the problem of the torsion of a simply connected prismatic body when there is a constant dislocation density ($\beta = \beta_0 = \text{const}$) and when there is no external torque. By relations (5.3), (2.1) and (2.5) this problem has a unique solution $F = 0$, $\omega = -\beta_0/2$. Hence, a uniform distribution of screw dislocations does not create stresses in a rod with a simply connected cross-section, although it also gives rise to torsion. This result is obvious in the light of the membrane analogy.

6. GENERAL THEOREMS OF THE THEORY OF THE TORSION OF RODS CONTAINING DISLOCATIONS

Suppose $F(x_1, x_2)$ is the solution of boundary-value problem (5.3), (2.1), (2.4) for a specified angle of torsion ω, a specified dislocation density $\beta(x_1, x_2)$ and given Volterra dislocation characteristics $b_t (t = 1, \ldots, N)$. Multiplying Eq. (5.3) by the function F, integrating over the region σ and using the divergence theorem, we arrive at a formula which expresses a Clapeyron-type theorem (everywhere henceforth the integration is carried out over the region σ)

$$\iint V d\sigma = \frac{1}{2} M\omega + \frac{1}{2} \sum_{t=1}^{N} b_t C_t + \frac{1}{2} \iint \beta F d\sigma \tag{6.1}$$

Using (6.1) and the fact that the function V is positive we can prove a theorem of uniqueness of the solution of the problem of the torsion of an elastic body with dislocations. The theorem of uniqueness also remains true for a problem with an unknown angle of torsion ω, but with a specified torque M.

We will consider two solutions of the torsion problem F' and F'', i.e. two equilibrium states of the bar, corresponding to two systems of external conditions β', ω' and b'_t and β'', ω'' and b''_t, respectively. Using the fact that the tensor $\mathbf{L}$ possesses the property of symmetry, we can prove the following reciprocity theorem

$$M'\omega'' + \sum_{t=1}^{N} C'_t b''_t + \iint F'\beta'' d\sigma = M''\omega' + \sum_{t=1}^{N} C''_t b'_t + \iint F''\beta' d\sigma \tag{6.2}$$

As an example of the application of the reciprocity theorem, consider the problem of determining the angle of torsion of a bar of simply connected cross-section for a known dislocation distribution and zero torque. This equilibrium state, which we will set to be the first, is described by the following boundary-value problem

$$\operatorname{div}(\mathbf{L} \cdot \operatorname{grad} F') = -2\omega' - \beta'(x_1, x_2) \tag{6.3}$$
$$F'|_{\Gamma_0} = 0, \quad \iint F' d\sigma = 0$$

We will take as the second state of the bar, that which exists when there is a constant unit dislocation density ($\beta'' \equiv 1$) and no torsion ($\omega'' = 0$). This state is determined by solving the boundary-value problem

$$\operatorname{div}(\mathbf{L} \cdot \operatorname{grad} F'') = -1, \quad F''|_{\Gamma_0} = 0 \tag{6.4}$$

Using formula (6.2) we obtain an expression for the angle of torsion in terms of the solution of boundary-value problem (6.4)

$$\omega' = \left(-\iint \beta' F'' d\sigma\right) / \left(2 \iint F'' d\sigma\right) \tag{6.5}$$

Hence, to determine the torsion of the bar, due to the specified dislocation distribution, there is no need to solve boundary-value problem (6.3), which contains an arbitrary function on the right-hand side of the generalized Poisson's equation, and it is sufficient to solve the simpler standard torsion problem (6.4), for which the right-hand side of the generalized Poisson's equation is constant.

Using (6.5) it is easy to obtain the so-called Eshelby torsion [4, 5], which arises when there is a lumped dislocation at the point (ξ_1, ξ_2) with a length of the Burgers vector B

$$\omega' = -BF''(\xi_1, \xi_2) / \left(2 \iint F'' d\sigma\right) \tag{6.6}$$

In particular, for an orthotropic bar ($l = 0$), the cross-section of which is bounded by the ellipse $x_1^2/a^2 + x_2^2/b^2 = 1$, problem (6.4) has the following solution

$$F'' = \frac{a^2b^2}{2(mb^2 + ka^2)}\left(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}\right) \tag{6.7}$$

From relations (6.6) and (6.7) we obtain

$$|\omega'| = \frac{B}{\pi ab}\left(1 - \frac{\xi_1^2}{a^2} - \frac{\xi_2^2}{b^2}\right) \tag{6.8}$$

When $a = b$ this formula becomes the well-known expression [4] for the torsion of an isotropic circular cylinder with a lumped screw dislocation.

7. THE DISLOCATION ENERGY IN A ROD OF RECTANGULAR CROSS-SECTION

Consider the problem of the stressed state of a bar of rectangular cross-section $0 \leqslant x_1 \leqslant a$, $0 \leqslant x_2 \leqslant b$ containing a lumped screw dislocation at an arbitrary point (ξ_1, ξ_2). We will assume the material to be orthotropic ($l = 0$). From the physical point of view, it is of interest to investigate the energy of an elastic body due to a dislocation as a function of the location of the dislocation. Since the solution of Eq. (5.3) with a delta function on the right-hand side has a logarithmic singularity, the energy of the body with a lumped dislocation will be infinite. In order to eliminate this divergence of the energy, we will replace the lumped dislocation with a specified length of the Burgers vector B by a continuous dislocation distribution with a density which is constant inside the square $|x_1 - \xi_1| < d/2$, $|x_2 - \xi_2| < d/2$ and equal to B/d^2, and equal to zero outside this square. The side of the square d is extremely small compared with the dimensions of the cross-section.

Thus, the function β on the right-hand side of Eq. (5.3) is taken in the form

$$\beta(x_1, x_2) = \begin{cases} B/d^2, & |x_1 - \xi_1| < d/2, \quad |x_2 - \xi_2| < d/2 \\ 0, & |x_1 - \xi_1| > d/2, \quad |x_2 - \xi_2| > d/2 \end{cases} \tag{7.1}$$

The solution of boundary-value problem (2.1), (5.3), (7.1) will be found in the form of double trigonometric series

$$F(x_1, x_2) = \frac{4a^2}{m\pi^2}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{1}{(s^2 + \alpha t^2)st}\left(\frac{B}{d^2}\Psi_{st} + 2\omega\lambda_{st}\right)\sin\left(\frac{\pi s}{a}x_1\right)\sin\left(\frac{\pi t}{b}x_2\right) \tag{7.2}$$

$$\Psi_{st} = \sin\left(\frac{\pi s}{2a}d\right)\sin\left(\frac{\pi t}{2b}d\right)\sin\left(\frac{\pi s}{a}\xi_1\right)\sin\left(\frac{\pi t}{b}\xi_2\right), \quad \alpha = \frac{ka^2}{mb^2}$$

$$\lambda_{st} = \frac{1}{4}(\cos\pi s - 1)(\cos\pi t - 1)$$

The Eshelby torsion is found from the condition $M = 0$ and has the form

$$\omega = -\frac{B}{2d^2}\left(\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{\Psi_{st}}{(s^2 + \alpha t^2)s^2t^2}\right)\times\left(\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{1}{(s^2 + \alpha t^2)s^2t^2}\right)^{-1} \tag{7.3}$$

The summation in (7.3) is only taken over odd s and t.

The dislocation energy is found from formula (6.1)

$$P(\xi_1, \xi_2) = -\frac{8a^2b}{m\pi^4}\frac{B}{d^2}\times\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{1}{(s^2 + \alpha t^2)s^2t^2}\left(\frac{B}{d^2}\Psi_{st} + 2\omega\lambda_{st}\right)\Psi_{st} \tag{7.4}$$

The value of ω is found from relation (7.3).

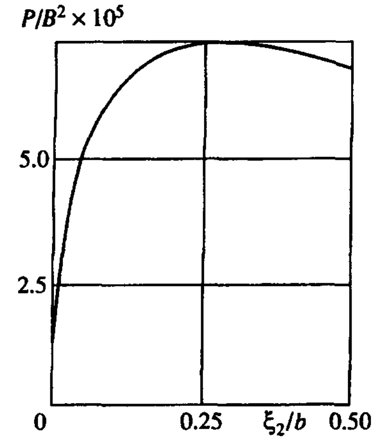

Fig. 2

In Fig. 2 we show schematically the energy P as a function of the position of the dislocation, i.e. as a function of the coordinates ξ_1, $\xi_2 (0 \leqslant \xi_1 \leqslant a,\ 0 \leqslant \xi_2 \leqslant b)$. The curve in Fig. 2 is a section of the surface $P(\xi_1, \xi_2)$ by the plane $\xi_1 = a/2$. In view of the symmetry of this section about the vertical axis we only show part of the graph, when ξ_2 varies from 0 to $b/2$.

Detailed numerical calculations showed that the sections of the surface $P(\xi_1, \xi_2)$ by the planes $\xi_1 = c\xi_2$ for any c have a minimum at the centre of the region. This means that the point $\xi_1 = \xi_2 = 0$ is a minimum of the function $P(\xi_1, \xi_2)$. No other minima were found in the numerical analysis. This enables us to assert that the central position of the dislocation is stable. Note that the potential well described exists for any ratio of the compliances k and m and any ratio of the sides of the rectangle.

Since, as was shown above, $d/a \ll 1$, in the calculations d was varied from $d = 10^{-6}a$ to $d = 10^{-4}a$. For these variations of the constant d the results did not qualitatively change. The value of the constant B plays no part since it is in fact the function $P(\xi_1, \xi_2)/B^2$ that was investigated. Stable results are obtained when the first 10^4 terms in series (7.2)–(7.4) are taken into account.

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